A GENERALISED MATHEMATICAL FORMULA FOR CONSTRUCTING VOLUME TABLES

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The construction of volume tables by a purely mathematical process has the advantage of eliminating to a large extent the personal factor which is present in customary graphic methods and can, with practice, provide a rapid means of analysing the basic data. Volume tables which have been prepared for exotic species by the method outlined in this paper have proved to be well within the limits of accuracy demanded by the New Zealand Forest Service, and the method has also been checked for its applicability to the measurement of indigenous species. The formula used is :

$$V = \left[D + \left(\frac{D}{C_1}\right)^2\right]\left[\frac{H}{C_2} \pm \left(\frac{H}{C_3}\right)^2\right]$$

where V=volume
D=diameter at breast height (over bark)
H=height
and C_1, C_2 and C_3 are constants.

Object and Scope

The object of this formula is to reproduce in graph form the characteristatics of any type of volume table. Thus the formula is not an attempt to interpret mathematically tree form or any other specific characteristics, but it simply aims at "plasticity," so that alterations of the constants alone shall reproduce those variations of slope, spacing and pitch of the curves which are charactersitic of various species and localities. The manner in which the constants affect the trends of the curves can be summarised as follows :

C_1 determines the degree of curvature of the height class curves or, using the metaphor of a suspended wire, the amount of "sag" of these curves.

C_2 determines the equal spacing between adjacent height class curves.

C_3 determines the degree with which such height class curves become more closely or widely spaced with increasing height values.

The combined effect of C_2 and C_3 is to determine, in their function as a common divisor of the first bracket, the over-all slope of the height curves.

The term $\sqrt{D}$ has been included in the first bracket in order to allow for bark volume. Bark thickness is not proportional to diameter and the use of this term provides a means of adjusting the curves in a manner similar to the effect which varying bark thickness has on their form.

It should also be noted that the formula gives zero values when, either D or H are zero.

Arrangement of the Basic Data

The basic data are first arranged into ten-foot height classes, and one inch d.b.h. classes within those height classes. These are then plotted by diameter classes and rough curves drawn, ensuring that they originate from zero. This gives an indication of the accuracy of the basic data which is of value in plotting by height classes, which is the next stage. Neither set of curves need be constructed with any great accuracy as they are simply analyses to obtain from the basic data one d.b.h. and one height class curve which, either by reason of its general relationship to the other curves plotted or, as frequently happens, because it contains the greatest number of trees, can be accepted as being sufficiently accurate to commence the mathematical analysis which follows.

Determination of the Constants

(a) From the chosen height class curve two values of d.b.h. are selected as far apart as possible. Then, if these diameters are called D_1 and D_2, and the corresponding volumes are V_1 and V_2, by substituting in the formula :

$$V_1 = \left[D_1 + \frac{\sqrt{D_1}}{C_1}\right]^2 \left[\frac{H}{C_2} + \frac{H^2}{C_3}\right] \quad \ldots\ldots\ldots\ldots \quad (a)$$

and $$V_2 = \left[D_2 + \frac{\sqrt{D_2}}{C_1}\right]^2 \left[\frac{H}{C_2} + \frac{H^2}{C_3}\right] \quad \ldots\ldots\ldots\ldots \quad (b)$$

Dividing $\dfrac{(a)}{(b)}$ $$\frac{V_1}{V_2} = \frac{\left[D_1 + \dfrac{\sqrt{D_1}}{C_1}\right]^2}{\left[D_2 + \dfrac{\sqrt{D_2}}{C_1}\right]^2}$$

This can be re-arranged into the form

$$xC_1^2 + 2yC_1 + z = 0$$

and C_1 can be determined since

$$C_1 = \frac{-y \pm \sqrt{y^2 - 4xz}}{2x}$$

(b) C_2 and C_3 are best determined from the chosen d.b.h. class curve. Examination of this curve gives an indication of the value and nature of $\frac{H^2}{C_3}$. If it is a straight line passing through the origin, the second bracket will contain only the factor $\frac{H}{C_2}$. On the other hand if it is an ascending curve the second bracket will contain $\frac{H}{C_2} + \frac{H^2}{C_3}$. Where H represents merchantable height as in the volume tables for our indigenous species, a third type of curve is found, the slope of which is less with increasing height values, and in this case the terms $\frac{H}{C_2} - \frac{H^2}{C_3}$ must be used. As already stated the second bracket controls the spacing of the height class curves, and the combinations of terms used above have the following effects on these curves.

$\left[D + \sqrt{\frac{D}{C_1}}\right]^2 \frac{H}{C_2}$ produces equally spaced height class curves.

$\left[D + \sqrt{\frac{D}{C_1}}\right]^2 \left[\frac{H}{C_2} + \frac{H}{C_3}\right]^2$ causes the curves to be increasingly widely spaced with increasing height values.

$\left[D + \sqrt{\frac{D}{C_1}}\right]^2 \left[\frac{H}{C_2} - \frac{H}{C_3}\right]^2$ results in curves increasingly more closely spaced with increasing height values.

In either case the straight line or the tangents to the d.b.h. curves at the origin represent :

$$V = \frac{H}{C_2}\left[D + \sqrt{\frac{D}{C_1}}\right]^2$$

but the values of V, H, D, and C_1 for any given point are known, and therefore by substituting these values, the value of C_2 can be determined.

(c) Similarly C_3 is determined by selecting any particularly well defined point on a curve and substituting in the formula the known values of V, H, D, C_1 and C_2.

Checking and Adjustment of Constants

A full volume table using the values determined above can be rapidly constructed by using a slide rule to determine a series of values for the first and second brackets separately ; these are then multiplied together on the slide rule for all the required combinations of d.b.h. and height values.

A first check should then be carried out by d.b.h./height classes without interpolation for individual trees. Alterations to the constants can be based on the following observations.

(a) Errors in the centre of height class curves, although the terminal values are correct, requires an adjustment of C_1, and C_2 and C_3 must be re-determined.

(b) Where the greater height classes are markedly incorrect but the lesser classes not so inaccurate, C_3 should be altered and C_2 re-adjusted.

(c) Where there is an over-all tendency for height values to be incorrect, i.e. the spacing is incorrect but the curvature is apparently accurate, C_2 should be treated as a simple divisor, and can be altered to change the value of the whole by the percentage determined in the check without interpolation. If the initial analyses have been made correctly this is the most likely source of error, and final adjustments to reduce the aggregate difference are very readily effected by minor alterations of C_2.

(d) A final check with interpolation for individual trees must be made, and here again the general trend of the curves should be found to be correct, so that an adjustment to C_2 on the basis of the percentage error only should be necessary to produce a final table.

Summary

(1) It is believed that all the curve characteristics which are exhibited in volume tables can be reproduced by the generalised mathematical formula discussed in this paper by altering the values of one or more of the three constants involved.

(2) The method outlined attempts to eliminate the personal factor present in volume table construction by customary graphic methods, and to provide a rapid means of analysing the basic data.

(3) Volume tables already constructed by this method have proved to be well within the limits of accuracy required by the New Zealand Forest Service, and it is felt that if the curve characteristics are basically sound, the mathematical method provides an accurate means of extending the table beyond the basic data.